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SETS OF p -MULTIPLICITY IN LOCALLY COMPACT GROUPS

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ABSTRACT. We initiate the study of sets of p -multiplicity in locally compact groups and their operator versions. We show that a closed subset E of a second countable locally compact group G is a set of p -multiplicity if and only if the set $E^* = \{(s, t) : ts^{-1} \in E\}$ is a set of operator p -multiplicity. We exhibit examples of sets of p -multiplicity, establish preservation properties for unions and direct products, and prove a p -version of the Stone-von Neumann Theorem.

1. INTRODUCTION

The existence of non-zero compact operators acting on a Hilbert space and leaving invariant a given commutative subspace lattice was first examined in [10] (see also [6] and the references therein). That work followed W. B. Arveson's seminal paper [1], and showed that the presence of non-zero compact operators in CSL algebras is closely related to the notion of multiplicity sets in commutative Harmonic Analysis. This relation was formalised, and generalised to non-commutative locally compact groups, in [17], where the notion of sets of operator multiplicity was introduced, and [18], where it was shown that a closed subset E of a (second countable) locally compact group G is a set of multiplicity if and only if the set $E^* = \{(s, t) : ts^{-1} \in E\}$ is a set of operator multiplicity.

The study of non-zero operators from Schatten p -classes in CSL algebras was also initiated in [10], where a link between such operators and pseudomeasures on compact abelian groups, whose Fourier transforms belong to the sequence space ℓ^p , was exhibited. In the case ℓ^p is replaced by c_0 , this turns into a special case of the result described in the previous paragraph. It is thus natural to define and study sets of p -multiplicity, their operator analogues, and the relation between these two notions.

This is the aim of the present article. In Section 3, given a locally compact group G , we define a subspace $\mathcal{S}_p(G)$ of the reduced group C*-algebra $C_r^*(G)$ of G that plays a role analogous to the role of the Schatten p -class within the C*-algebra of all compact operators on a Hilbert space. In the case the group G is compact, the space $\mathcal{S}_p(G)$ coincides with the intersection of $C_r^*(G)$ with the Schatten p -class on $L^2(G)$. It should be noted that if G is discrete, $\mathcal{S}_p(G)$ is equal to $C_r^*(G)$, and thus the interest in our work lies in the case where G is locally compact and non-discrete; for example, in the case where G is connected. After defining sets of p -multiplicity and their

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operator versions, we show that a closed set $E \subseteq G$ is a set of p -multiplicity if and only if E^* is a set of operator p -multiplicity. We give a number of examples of sets of p -multiplicity, and establish preservation properties for unions and direct products. We include characterisations of the sets of p -multiplicity in the case $p = 1$ and $p = 2$.

In Section 4, we prove a p -version of the Stone-von Neumann Theorem. Recall that this result can be stated by saying that the C^* -algebra of all compact operators on $L^2(G)$ is generated by $C_r^*(G)$ and the multiplication algebra of the space $C_0(G)$ of all continuous functions on G vanishing at infinity. Here, we obtain an analogous result for the Schatten p -class, using the space $\mathcal{S}_p(G)$ in the place of $C_r^*(G)$.

In Section 5, using the Fourier theory of compact groups, we give a different proof of the aforementioned transference theorem for sets of p -multiplicity, which we believe is interesting in its own right.

Finally, in Section 2 we collect the necessary background material and set notation.

2. PRELIMINARIES

Let (X, μ) and (Y, ν) be standard (σ -finite) measure spaces. A subset $E \subseteq X \times Y$ is called *marginally null* if $E \subseteq (M \times Y) \cup (X \times N)$, where $M \subseteq X$ and $N \subseteq Y$ are null sets. Let $T(X, Y)$ be the projective tensor product $L^2(X) \hat{\otimes} L^2(Y)$. Every element $h \in T(X, Y)$ can be written as a series

$$h = \sum_{i=1}^{\infty} f_i \otimes g_i, \quad f_i \in L^2(X), g_i \in L^2(Y), i \in \mathbb{N},$$

where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. Such an element h may be considered either as a function $h : X \times Y \rightarrow \mathbb{C}$, defined up to a marginally null set and given by

$$h(x, y) = \sum_{i=1}^{\infty} f_i(x) g_i(y),$$

or as an element of the predual of the space $\mathcal{B}(L^2(X), L^2(Y))$ of all bounded linear operators from $L^2(X)$ into $L^2(Y)$ via the pairing

$$\langle T, h \rangle := \sum_{i=1}^{\infty} (T f_i, \bar{g}_i).$$

We denote by $\|h\|_T$ the norm of $h \in T(X, Y)$.

Let $\mathfrak{S}(X, Y)$ be the multiplier algebra of $T(X, Y)$; by definition, a measurable function $w : X \times Y \rightarrow \mathbb{C}$ belongs to $\mathfrak{S}(X, Y)$ if the map $m_w : h \rightarrow wh$ leaves $T(X, Y)$ invariant, that is, if wh coincides almost everywhere with a function from $T(X, Y)$, for every $h \in T(X, Y)$. The elements of $\mathfrak{S}(X, Y)$ are called (*measurable*) *Schur multipliers*; we refer the reader to [15] for relevant details. If $w \in \mathfrak{S}(X, Y)$, the adjoint of m_w , acting on $\mathcal{B}(L^2(X), L^2(Y))$, will be denoted by S_w .

Throughout the paper, G is a locally compact group. The Lebesgue spaces $L^p(G)$, $p = 1, 2, \infty$, are with respect to left Haar measure m ; $dm(x)$

is shortened to dx and the modular function of G is denoted by Δ . Let $\lambda : G \rightarrow \mathcal{B}(L^2(G))$, $s \mapsto \lambda_s$, be the left regular representation. The symbol λ is used also for the corresponding representation of $L^1(G)$ on $L^2(G)$; thus, if $f \in L^1(G)$ then $\lambda(f)$ is the operator on $L^2(G)$ given by $\lambda(f)(g) = f * g$.

The *reduced group C^* -algebra* $C_r^*(G)$ of G is the operator norm closure of $\{\lambda(f) : f \in L^1(G)\}$, while the *group von Neumann algebra* of G is its weak* closure. The *Fourier algebra* $A(G)$ of G is the (commutative, regular, semi-simple) Banach algebra consisting of all complex functions u on G of the form

$$(2.1) \quad x \rightarrow u(x) = (\lambda_x \xi, \eta),$$

where $\xi, \eta \in L^2(G)$. The norm of an element $u \in A(G)$ is by definition the infimum of the products $\|\xi\| \|\eta\|$, where ξ and η are functions from $L^2(G)$ for which (2.1) holds. The Banach space dual of $A(G)$ can be canonically identified with $\text{VN}(G)$: for $T \in \text{VN}(G)$ and u as in (2.1), the pairing is given by

$$\langle T, u \rangle = (T\xi, \eta);$$

we refer the reader to [8] for this and further properties of $A(G)$.

We set $T(G) = T(G, G)$, $\mathfrak{S}(G) = \mathfrak{S}(G, G)$ and $\mathcal{B}(L^2(G)) = \mathcal{B}(L^2(G), L^2(G))$. The map $P : T(G) \rightarrow A(G)$, given by

$$(2.2) \quad P(f \otimes g)(t) = \langle \lambda_t, f \otimes g \rangle = (\lambda_t f, \bar{g}) = \int_G f(t^{-1}s)g(s)ds = g * \check{f}(t)$$

(where $\check{f}(t) = f(t^{-1})$) is the predual of the inclusion $\text{VN}(G) \rightarrow \mathcal{B}(L^2(G))$. Moreover, the following holds (see [18] for a proof):

Proposition 2.1. *For every $h \in T(G)$, we have*

$$P(h)(t) = \int_G h(t^{-1}s, s)ds, \quad t \in G.$$

Define

$$N : L^\infty(G) \rightarrow L^\infty(G \times G) \quad \text{by} \quad N(f)(s, t) = f(ts^{-1}).$$

We will often use the fact that if $u \in A(G)$ then $N(u) \in \mathfrak{S}(G)$. More generally, the set of all continuous functions $u : G \rightarrow \mathbb{C}$ such that $N(u) \in \mathfrak{S}(G)$ coincides with the algebra $M^{\text{cb}}A(G)$ of all *completely bounded*, or *Herz-Schur multipliers* of $A(G)$ [3], [19] (see also [11]). For $v \in A(G)$ and $T \in \text{VN}(G)$, let $v \cdot T \in \text{VN}(G)$ be the element of $\text{VN}(G)$ given by

$$\langle v \cdot T, u \rangle = \langle T, vu \rangle, \quad u \in A(G);$$

we have $v \cdot T = S_{N(v)}(T)$ (see, e.g., [13]).

We denote by $\mathcal{S}_p(H)$ the Schatten p -class on a Hilbert space H (here, $1 \leq p < \infty$), and we let $S_\infty(H)$ be the space of all compact operators on H . If H is clear from the context, we simply write \mathcal{S}_p . We write $\|T\|_p$ for the Schatten p -norm of an element $T \in \mathcal{S}_p$, $1 \leq p < \infty$, and let $\|T\|_\infty = \|T\|$ for the usual operator norm of an element $T \in \mathcal{S}_\infty$.

For a function $h \in L^2(G \times G)$, we let $T_h \in \mathcal{S}_2(L^2(G))$ be the operator given by

$$(2.3) \quad T_h(\xi)(y) = \int_G h(y, x)\xi(x)dx, \quad \xi \in L^2(G), y \in G.$$

We call the function h the *integral kernel* of T_h . We note that if $h \in T(G)$ then $T_h \in \mathcal{S}_1(L^2(G))$; conversely, for every operator $T \in \mathcal{S}_1(L^2(G))$ there exists $h \in T(G)$ such that $T = T_h$.

For a measure space (X, μ) , and a function $a \in L^\infty(X, \mu)$, we let M_a denote the (bounded) operator on $L^2(X)$ of multiplication by a and P_K the multiplication by the characteristic function χ_K of measurable subset $K \subseteq X$.

3. DEFINITIONS AND PROPERTIES

Let G be a locally compact group. For each $1 \leq p \leq \infty$, let

$$\mathcal{S}_p(G) = \{T \in C_r^*(G) : P_K T P_K \in \mathcal{S}_p, \text{ for all compact subsets } K \subseteq G\}.$$

Note first that, if $f \in C_c(G)$ then $P_K \lambda(f) P_K$ is an integral operator with integral kernel

$$(s, t) \mapsto \chi_{K \times K}(s, t) \Delta(t)^{-1} f(st^{-1}).$$

Thus, $P_K \lambda(f) P_K$ is a Hilbert-Schmidt operator. Since every $T \in C_r^*(G)$ can be approximated in the operator norm by operators of the form $\lambda(f)$ with $f \in C_c(G)$, we conclude that $P_K T P_K \in \mathcal{S}_\infty$ whenever $T \in C_r^*(G)$ and $K \subseteq G$ is compact; thus, $\mathcal{S}_\infty(G) = C_r^*(G)$.

Remarks (i) Let $v \in M^{\text{cb}}A(G)$ and $T \in \mathcal{S}_p(G)$. Then $v \cdot T \in \mathcal{S}_p(G)$. Indeed, for every compact set $K \subseteq G$, we have that

$$P_K(v \cdot T)P_K = P_K S_{N(v)}(T)P_K = S_{N(v)}(P_K T P_K) \in \mathcal{S}_p,$$

since Schur multipliers leave \mathcal{S}_p invariant (the latter fact can be easily seen by using a complex interpolation argument, see [2, 16] and the proof of Theorem 3.3).

(ii) If $p \leq q$ then $\mathcal{S}_p(G) \subseteq \mathcal{S}_q(G)$.

(iii) If G is discrete, $K \subseteq G$ is compact precisely when it is finite; thus, in this case, $\mathcal{S}_p(G) = C_r^*(G)$ for all values of p .

(iv) If G is compact then $\mathcal{S}_p(G) = C_r^*(G) \cap \mathcal{S}_p$. Indeed, the inclusion $\mathcal{S}_p \cap C_r^*(G) \subseteq \mathcal{S}_p(G)$ holds trivially for any G . If G is compact and $T \in \mathcal{S}_p(G)$ then, taking $K = G$ in the definition of $\mathcal{S}_p(G)$, we see that $T \in \mathcal{S}_p$.

In case G is compact, the previous paragraph shows that $\mathcal{S}_p(G)$ is an ideal of $C_r^*(G)$. Moreover, the inclusion $\mathcal{S}_p(G) \subseteq \mathcal{S}_q(G)$, $p < q$, is proper if G is infinite (see Remark 5.2). We do not know whether the spaces $\mathcal{S}_p(G)$ are ideals for other classes of locally compact groups G .

(v) The identity from Remark (iv) fails when G is not compact. Indeed, it is known (see, e.g., [20]) that in this case $\text{VN}(G) \cap \mathcal{S}_\infty = \{0\}$.

(vi) Since for any compact subset $K \subseteq G$ and $f \in C_c(G)$ the operator $P_K \lambda(f) P_K$ is Hilbert-Schmidt, $\lambda(C_c(G)) \subseteq \mathcal{S}_2(G)$.

(vii) Let G be a compact abelian group and \widehat{G} be its dual group. Then $\text{VN}(G)$ and $C_r^*(G)$ can be identified, *via* Fourier transform, with the spaces $\ell^\infty(\widehat{G})$ and $c_0(\widehat{G})$. It is easily seen that, under this identification, $\mathcal{S}_p(G)$ is sent onto the sequence space $\ell_p(\widehat{G})$. Thus, in this case we have $\mathcal{S}_p(G) = \text{VN}(G) \cap \mathcal{S}_p = C_r^*(G) \cap \mathcal{S}_p$.

(viii) Let $G = \mathbb{R}$ and $L \subseteq \widehat{\mathbb{R}} = \mathbb{R}$ be a compact interval. Then $\chi_L \in L^\infty(\widehat{\mathbb{R}}) \setminus C_0(\widehat{\mathbb{R}})$. Thus, if T is the operator in $\text{VN}(\mathbb{R})$ corresponding to χ_L *via* Fourier transform, then $T \notin C_r^*(\mathbb{R})$. However, if $K \subseteq \mathbb{R}$ is compact then the operator $P_K T P_K$ is easily seen to be an integral operator with integral kernel

$$(s, t) \mapsto \chi_{K \times K}(s, t) \int_L e^{-ix(s-t)} dx.$$

Since $|\int_L e^{-ix(s-t)} dx| \leq m(L)$ for all $s, t \in \mathbb{R}$, the operator $P_K T P_K$ belongs to \mathcal{S}_2 . This example shows that, in contrast to the compact case, replacing $C_r^*(G)$ by $\text{VN}(G)$ in the definition of $\mathcal{S}_p(G)$ will in general yield different spaces.

Recall that, given a closed subset $E \subseteq G$, $I(E)$ (resp. $J(E)$) is the largest (resp. the smallest) ideal of $A(G)$ with null set E :

$$I(E) = \{u \in A(G) : u(s) = 0, s \in E\}$$

and

$$J(E) = \overline{\{u \in A(G) : u \text{ has compact support disjoint from } E\}}.$$

For $J \subseteq A(G)$ we denote by J^\perp the annihilator of J in $\text{VN}(G)$.

Sets of multiplicity (or M -sets) in (general) locally compact groups were introduced in [4] (see also [5]), while in [18], the notion of M_1 -set was defined. We next formulate p -versions of these concepts as follows.

Definition 3.1. A closed subset $E \subseteq G$ will be called

- (i) an M^p -set (or a *set of p -multiplicity*) if $J(E)^\perp \cap \mathcal{S}_p(G) \neq \{0\}$;
- (ii) an M_1^p -set if $I(E)^\perp \cap \mathcal{S}_p(G) \neq \{0\}$.

It is clear that every M_1^p -set is an M^p -set, and that M_1^∞ -sets (resp. M^∞ -sets) coincide precisely with M_1 -sets (resp. M -sets) studied in [18].

Recall that the support $\text{supp}(T)$ of an operator $T \in \text{VN}(G)$ is defined by letting

$$\text{supp}(T) = \{t \in G : u \cdot T \neq 0 \text{ whenever } u \in A(G) \text{ and } u(t) \neq 0\}.$$

Note that $J(E)^\perp$ coincides with the space of all operators $T \in \text{VN}(G)$ for which $\text{supp}(T) \subseteq E$. Hence a subset E is an M^p -set if and only if there exists a non-zero operator T in $\mathcal{S}_p(G)$ with $\text{supp}(T) \subseteq E$.

Our next aim is to define operator versions of sets of p -multiplicity. We first recall some concepts from [1] and [7]. Given standard measure spaces (X, μ) and (Y, ν) , a subset E of $X \times Y$ is called ω -open if it is marginally equivalent to the union of a countable set of Borel rectangles. The complements of ω -open sets are called ω -closed. A function $w : X \times Y \rightarrow \mathbb{C}$ is called ω -continuous if $w^{-1}(U)$ is an ω -open set for every open set $U \subseteq \mathbb{C}$.

If $F \subseteq X \times Y$ is an ω -closed set, an operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is said to be *supported on F* if

$$(A \times B) \cap F \simeq \emptyset \implies P_B T P_A = 0,$$

for all measurable rectangles $A \times B \subseteq X \times Y$. A masa-bimodule is a subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$ such that $\mathcal{D}_Y \mathcal{U} \mathcal{D}_X \subseteq \mathcal{U}$ (where \mathcal{D}_X (resp. \mathcal{D}_Y) is the multiplication masa of $L^\infty(X)$ (resp. $L^\infty(Y)$)). Given a masa-bimodule \mathcal{U} , there exists a smallest, up to marginal equivalence, ω -closed subset $\kappa \subseteq X \times Y$ such that every operator in \mathcal{U} is supported by F ; we call F the *support* of \mathcal{U} . Given an ω -closed set $\kappa \subseteq X \times Y$, there exist [1], [7] a largest weak* closed masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ and a smallest weak* closed masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ with support κ . The masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ is the space of all $T \in \mathcal{B}(L^2(X), L^2(Y))$ supported on κ .

Definition 3.2. An ω -closed subset $\kappa \subseteq X \times Y$ will be called

- (i) an *operator M^p -set* (or a *set of operator p -multiplicity*) if $\mathfrak{M}_{\max}(\kappa) \cap \mathcal{S}_p \neq \{0\}$;
- (ii) an *operator M_1^p -set* if $\mathfrak{M}_{\min}(\kappa) \cap \mathcal{S}_p \neq \{0\}$.

Remarks (i) Note that, if $1 \leq p \leq 2$ then the two notions introduced in Definition 3.2 agree; this follows from the fact that every Hilbert-Schmidt operator is *pseudo-integral*, while the pseudo-integral operators supported on a subset κ are contained (in fact, weak* dense) in $\mathfrak{M}_{\min}(\kappa)$. We refer the reader to [1] for the definition and more details about the class of pseudo-integral operators.

(ii) A subset $\kappa \subseteq X \times Y$ is an operator M^2 -set if and only if $(\mu \times \nu)(\kappa) > 0$; indeed, the latter condition is equivalent to the existence of non-zero functions $h \in L^2(X \times Y)$ supported on κ .

In [18] we established a connection between sets of multiplicity and sets of operator multiplicity. The next theorem is a generalisation of this result to sets of p -multiplicity. For $\varphi \in T(G)$, let $E_\varphi : \mathcal{B}(L^2(G)) \rightarrow VN(G)$ be the map given by

$$\langle E_\varphi(T), u \rangle = \langle T, \varphi N(u) \rangle, \quad T \in \mathcal{B}(L^2(G)), u \in A(G),$$

where the pairing on the left hand side is the one between $VN(G)$ and $A(G)$, and on the right hand side – the one between $\mathcal{B}(L^2(G))$ and $T(G)$. It was proved in [18, Theorem 3.8] that $E_\varphi(T) \in C_r^*(G)$ for any $\varphi \in T(G)$ whenever T is compact.

For $E \subseteq G$, we let $E^* = \{(s, t) : ts^{-1} \in E\} \subseteq G \times G$. We will assume, for the rest of the paper, that G is *second countable*.

Theorem 3.3. *Let G be a locally compact group, $E \subseteq G$ be a closed subset and $p \geq 1$. The following are equivalent:*

- (i) *E is an M^p -set (resp. an M_1^p -set);*
- (ii) *E^* is an operator M^p -set (resp. an operator M_1^p -set).*

Proof. (i) \implies (ii) Suppose that $E \subseteq G$ is an M^p -set and let T be a non-zero operator in $J(E)^\perp \cap \mathcal{S}_p(G)$. Then there exists a compact set $K \subseteq G$ such that $P_K T P_K$ is non-zero; by [18, Lemma 3.11], $P_K T P_K \in \mathfrak{M}_{\max}(E^*) \cap \mathcal{S}_p$.

Let $E \subseteq G$ be an M_1^p -set. The proof of [18, Theorem 3.12 (b)] shows that $I(E)^\perp \subseteq \mathfrak{M}_{\min}(E^*)$. As in the previous paragraph, one can find a non-zero operator in $\mathfrak{M}_{\min}(E^*) \cap \mathcal{S}_p$.

(ii) \Rightarrow (i) Assume that E^* is an operator M^p -set; we will show that E is an M^p -set. If $p = \infty$, this follows from [18, Theorem 3.11]. Let $p = 1$ and T be a non-zero trace class operator in $\mathfrak{M}_{\max}(E^*)$; by virtue of (2.3) and the remark following it, write $T = T_h$, where $h = \sum_{i=1}^\infty f_i \otimes g_i$, with $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$.

Fix $\varphi \in T(G) \cap \mathfrak{S}(G)$ such that the function $\psi = \varphi(1 \otimes \Delta)$ belongs to $\mathfrak{S}(G)$. We will show that $E_\varphi(T) \in \mathcal{S}_1(G)$. For every $u \in A(G)$, we have

$$\begin{aligned} \langle E_\varphi(T), u \rangle &= \langle T, \varphi N(u) \rangle = \iint_{G \times G} h(s, t) \varphi(s, t) u(ts^{-1}) ds dt \\ &= \iint_{G \times G} h(r^{-1}t, t) \varphi(r^{-1}t, t) u(r) \Delta(tr^{-1}) dr dt \\ &= \int_G \left(\int_G h(r^{-1}t, t) \varphi(r^{-1}t, t) \Delta(t) dt \right) u(r) \Delta(r^{-1}) dr. \end{aligned}$$

By assumption, $\psi \in \mathfrak{S}(G)$ and hence $\psi h \in T(G)$. By Proposition 2.1,

$$P(\psi h)(r) = \int_G h(r^{-1}t, t) \varphi(r^{-1}t, t) \Delta(t) dt$$

and hence

$$(3.1) \quad \langle E_\varphi(T), u \rangle = \int_G P(\psi h)(r) u(r) \Delta(r^{-1}) dr.$$

Let $\xi, \eta \in L^2(G)$ be such that $u(r) = (\lambda_r(\xi), \eta)$, $r \in G$. Then, by (3.1),

$$\begin{aligned} (E_\varphi(T)\xi, \eta) &= \langle E_\varphi(T), u \rangle = \iint_{G \times G} P(\psi h)(r) \Delta(r^{-1}) \xi(r^{-1}x) \overline{\eta(x)} dr dx \\ (3.2) \quad &= \iint_{G \times G} P(\psi h)(xy^{-1}) \Delta(yx^{-1}) \xi(y) \overline{\eta(x)} \Delta(y^{-1}) dy dx \\ &= \iint_{G \times G} P(\psi h)(xy^{-1}) \Delta(x^{-1}) \xi(y) \overline{\eta(x)} dy dx. \end{aligned}$$

Let

$$(3.3) \quad w(x, y) = P(\psi h)(xy^{-1}) \Delta(x^{-1}), \quad x, y \in G.$$

Identity (3.2) shows that w is an integral kernel and $E_\varphi(T) = T_w$. If $K \subseteq G$ is compact then $P_K T_w P_K = T_{w\chi_{K \times K}}$ and

$$(3.4) \quad w\chi_{K \times K} = \hat{N}(P(\psi h))((\Delta^{-1}\chi_K) \otimes \chi_K),$$

where $\hat{N}(v)(s, t) = v(st^{-1})$, $s, t \in G$. We have that $P(\psi h) \in A(G)$; thus $N(P(\psi h)) \in \mathfrak{S}(G)$ and hence $\hat{N}(P(\psi h)) \in \mathfrak{S}(G)$. Since $(\Delta^{-1}\chi_K) \otimes \chi_K \in T(G)$, identity (3.4) shows that $w\chi_{K \times K} \in T(G)$ and hence $P_K E_\varphi(T) P_K \in \mathcal{S}_1$. Thus, $E_\varphi(T) \in \mathcal{S}_1(G)$.

By [18, Lemma 3.10], there exist elements $c, d \in L^2(G)$ such that $E_{c \otimes d}(T) \neq 0$. Since the space \mathcal{F} of all compactly supported functions in $L^\infty(G)$ is dense in $L^2(G)$, the continuity of the map $\varphi \rightarrow E_\varphi(T)$ and [18, Proposition

3.8] imply that we may choose c and d from \mathcal{F} . However, in this case $d\Delta \in L^\infty(G)$ and hence $(c \otimes d)(1 \otimes \Delta) \in \mathfrak{S}(G)$. Letting $\varphi = c \otimes d$, we then have by the previous paragraphs that $E_\varphi(T) \neq 0$, and by the proof of [18, Theorem 3.11] that $E_\varphi(T) \in J(E)^\perp$. It follows that E is an M^1 -set.

To prove the statement for an arbitrary p , we use complex interpolation. Recall [16] that $(\mathcal{S}_1, \mathcal{S}_\infty)$ is a compatible couple, and \mathcal{S}_p coincides with the interpolation space between \mathcal{S}_1 and \mathcal{S}_∞ with parameter $\theta = p^{-1}$. Let, as above, $\varphi \in T(G) \cap \mathfrak{S}(G)$ be such that the function $\psi = \varphi(1 \otimes \Delta)$ is an element of $\mathfrak{S}(G)$. For a fixed compact set $K \subseteq G$ and $p = 1, \infty$, let $\Phi_p : \mathcal{S}_p \rightarrow \mathcal{B}(L^2(G))$ be the operator given by

$$\Phi_p(T) = P_K E_\varphi(T) P_K, \quad T \in \mathcal{S}_p.$$

By the previous paragraphs, the image of Φ_1 is in \mathcal{S}_1 . Moreover,

$$\begin{aligned} \|\Phi_1(T)\|_{\mathcal{S}_1} &= \|\chi_{K \times K} w\|_{T(G)} = \|\hat{N}(P(\psi h))((\Delta^{-1} \chi_K) \otimes \chi_K)\|_{T(G)} \\ &\leq \|\hat{N}(P(\psi h))\|_{\mathfrak{S}(G)} \|((\Delta^{-1} \chi_K) \otimes \chi_K)\|_{T(G)} \\ &\leq \|P(\psi h)\|_{A(G)} \|((\Delta^{-1} \chi_K) \otimes \chi_K)\|_{T(G)} \\ &\leq \|\psi h\|_{T(G)} \|((\Delta^{-1} \chi_K) \otimes \chi_K)\|_{T(G)} \\ &\leq m(K) \|\psi\|_{\mathfrak{S}(G)} \|h\|_{T(G)} \|\Delta^{-1} \chi_K\|_\infty \\ &= m(K) \|\psi\|_{\mathfrak{S}(G)} \|\Delta^{-1} \chi_K\|_\infty \|T\|_{\mathcal{S}_1}, \end{aligned}$$

which shows that the operator $\Phi_1 : \mathcal{S}_1 \rightarrow \mathcal{S}_1$ is bounded. On the other hand, the image of Φ_∞ is in \mathcal{S}_∞ and, by [18, Theorem 3.8],

$$\|\Phi_\infty(T)\| \leq \|\varphi\|_{T(G)} \|T\|, \quad T \in \mathcal{S}_\infty.$$

By complex interpolation, the image of the operator Φ_p is in \mathcal{S}_p . The proof is now completed by choosing φ for which $E_\varphi(T)$ is non-zero.

We have thus shown that if E^* is an operator M^p -set then E is an M^p -set. The proof of the case where E^* is an operator M_1^p -set follows similar arguments and uses the fact that $E_\varphi(T)$ belongs to $I(E)^\perp$ if $T \in \mathfrak{M}_{\min}(E^*)$ (see [18, Theorem 3.11]). \square

Corollary 3.4. *A closed subset E of a locally compact group G is an M^1 -set if and only if it has a non-empty interior.*

Proof. Suppose that E is an M^1 -set. By Theorem 3.3, $\mathfrak{M}_{\max}(E^*)$ contains a non-zero trace class operator; by [7, Theorem 6.7], E^* contains a non-trivial measurable rectangle, say, $\alpha \times \beta$. Thus $\beta\alpha^{-1} \subseteq E$. By Steinhaus' Theorem, $\beta\alpha^{-1}$, and hence E , has a non-empty interior.

Conversely, assume that U is an open subset of E ; we may further assume that U has a compact closure contained in E . Let $u \in A(G)$ be a function supported in U ; then $u \in L^1(G)$ and thus $\lambda(u) \in C_r^*(G)$. It is easy to see that $\lambda(u) \in J(E)^\perp$. Let $K \subseteq G$ be a compact set. Then $P_K \lambda(u) P_K$ is an integral operator with integral kernel $(t, s) \rightarrow u(ts^{-1}) \chi_K(t) \chi_K(s) \Delta(s)^{-1}$. The function $(t, s) \mapsto \chi_K(t) \chi_K(s) \Delta(s)^{-1}$ belongs to $T(G)$ since χ_K is compactly supported and Δ is continuous. Since $N(u) \in \mathfrak{S}(G)$, we conclude that the function $(t, s) \mapsto u(ts^{-1}) \chi_K(t) \chi_K(s) \Delta(s)^{-1}$ belongs to $T(G)$; since this holds for all compact sets K , we conclude that $\lambda(u) \in \mathcal{S}_1(G)$. \square

Corollary 3.5. *A closed subset E of a locally compact group G is an M^2 -set if and only if it has positive Haar measure.*

Proof. Note that $m(E) > 0$ if and only if $m \times m(E^*) > 0$. The claim follows from Theorem 3.3 and Remark (ii) after Definition 3.2. \square

We next include some examples.

Examples (i) J. Froelich [10, p. 13-14] has shown that there exists a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero which supports a non-zero measure μ whose Fourier transform vanishes at infinity but which does not support a non-zero pseudomeasure with Fourier transform in ℓ^p . The set E is an M -set that is not an M^p -set for any $p \geq 1$.

(ii) Let \mathbb{T} be the group of the unit circle, realised additively as $\mathbb{R}/2\pi\mathbb{Z}$. We identify \mathbb{T}^n with $[-\pi, \pi)^n$, and view the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ as a subset of \mathbb{T}^n . Let μ be the normalised surface area measure of S^{n-1} . A direct calculation (see, e.g., [21, p.154]) shows that $\hat{\mu}(k) = C|k|^{-(n-2)/2} J_{(n-2)/2}(|k|)$ for some constant C , where $\hat{\mu}$ is the Fourier transform of μ and $J_{(n-2)/2}$ is a Bessel function. As $|J_\nu(r)| \leq C_\nu r^{-1/2}$ (see [9, Theorem 5.1]), for large enough $r > 0$, we obtain $\hat{\mu}(k) = O\left(\frac{1}{|k|^{(n-1)/2}}\right)$, as $|k| \rightarrow \infty$.

We have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} |\hat{\mu}(k)|^p &\leq C \sum_{k \in \mathbb{Z}^n, |k| \geq 1} \frac{1}{|k|^{p(n-1)/2}} \leq C \int_{x \in \mathbb{R}^n, |x| \geq 1} \frac{1}{|x|^{(n-1)p/2}} dx \\ &= \int_1^\infty \frac{r^{n-1}}{r^{(n-1)p/2}} dr. \end{aligned}$$

Therefore for $p > 2$ and $n > 1 + \frac{2}{p-2}$, the sequence $\{\hat{\mu}(k)\}_{k \in \mathbb{Z}^n}$ belongs to $\ell^p(\mathbb{Z}^n)$ and hence $\lambda(\mu) \in \mathcal{S}_p(\mathbb{T}^n)$. As $\lambda(\mu) \in I(S^{n-1})^\perp$, we conclude that $S^{n-1} \subseteq \mathbb{T}^n$ is an M_1^p -set for $p > 2$ whenever $n > 1 + \frac{2}{p-2}$. Note that, by Corollary 3.5, S^{n-1} , $n > 1$, is not an M_1^2 -set since S^{n-1} has zero Lebesgue measure (see Corollary 3.5).

(iii) There exists a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and a non-zero measure μ with $\text{supp } \mu = E$ such that $\hat{\mu} \in \ell^p$ for any $p > 2$ (see [23, Theorem 10.12]). Hence E is an M_1^p -set for all $p > 2$ but not an M^2 -set (see Corollary 3.5).

(iv) In [7, p. 579] an example is given of a set $E \subseteq \mathbb{T}$ and a function $f \in L^2(\mathbb{T})$ such that $\lambda(f)$ is supported in E and $\lambda(f) \in \mathcal{S}_p$ for any $p > 1$ but $\lambda(f) \notin \mathcal{S}_1$. Remark (iv) from the start of Section 3 implies that E is an M^p -set for all $p > 1$ but not an M^1 -set.

Following the established terminology in the classical case, let us call a closed subset $E \subseteq G$ a set of p -uniqueness if it is not a set of p -multiplicity. In the remainder of this section, we apply Theorem 3.3 to establish some preservation results for sets of p -uniqueness and sets of p -multiplicity.

Proposition 3.6. *Let G be a locally compact group, $1 \leq p \leq \infty$ and $E_i \subseteq G$ be a closed subset, $i = 1, 2$. Then $E_1 \cup E_2$ is a set of p -uniqueness if and only if E_1 and E_2 are sets of p -uniqueness.*

Proof. To see the "if" part of the statement, it suffices, by Theorem 3.3, to show that if $\mathfrak{M}_{\max}(E_i^*) \cap \mathcal{S}_p = \{0\}$ for $i = 1, 2$, then $\mathfrak{M}_{\max}((E_1 \cup E_2)^*) \cap \mathcal{S}_p = \{0\}$.

Let $D_i = \mathfrak{M}_{\max}(E_i^*)_{\perp}$, $i = 1, 2$, and suppose that $T \in \mathfrak{M}_{\max}((E_1 \cup E_2)^*) \cap \mathcal{S}_p$. For each $\theta_i \in D_i \cap \mathfrak{S}(G)$, $i = 1, 2$, we have that $\theta_1 \theta_2 \in D_1 \cap D_2$ and, since $D_1 \cap D_2 = \mathfrak{M}_{\max}((E_1 \cup E_2)^*)_{\perp}$, we conclude that

$$\langle S_{\theta_1}(T), \theta_2 \rangle = \langle T, \theta_1 \theta_2 \rangle = 0.$$

However, Schur multipliers leave \mathcal{S}_p invariant; since $D_2 \cap \mathfrak{S}(G)$ is dense in D_2 and E_2 is a set of p -uniqueness, $S_{\theta_1}(T) = 0$. Thus, $\langle T, \theta_1 \rangle = 0$ for all $\theta_1 \in D_1 \cap \mathfrak{S}(G)$. Now the density of $D_1 \cap \mathfrak{S}(G)$ in D_1 and the fact that E_1 is a set of p -uniqueness imply that $T = 0$.

The 'only if' part of the statement follows from the fact that any closed subset of a set of p -uniqueness is a set of p -uniqueness. \square

Proposition 3.7. *Let G_i be a locally compact group, $1 \leq p \leq \infty$ and $E_i \subseteq G_i$ be a closed set, $i = 1, 2$. If E_i is an M^p -set (resp. M_1^p -set), $i = 1, 2$, then $E_1 \times E_2 \subseteq G_1 \times G_2$ is an M^p -set (resp. M_1^p -set).*

Proof. Let

$$\rho : G_1 \times G_1 \times G_2 \times G_2 \rightarrow G_1 \times G_2 \times G_1 \times G_2$$

be the map given by $\rho(s_1, t_1, s_2, t_2) = (s_1, s_2, t_1, t_2)$. We have that $(E_1 \times E_2)^* = \rho(E_1^* \times E_2^*)$.

By Theorem 3.3, it suffices to show that $\rho(E_1^* \times E_2^*)$ is an operator M^p -set (resp. an operator M_1^p -set). Denoting by \otimes the algebraic tensor product, we have

$$\mathfrak{M}_{\max}(E_1^*) \otimes \mathfrak{M}_{\max}(E_2^*) \subseteq \mathfrak{M}_{\max}(\rho(E_1^* \times E_2^*))$$

and

$$\mathfrak{M}_{\min}(E_1^*) \otimes \mathfrak{M}_{\min}(E_2^*) \subseteq \mathfrak{M}_{\min}(\rho(E_1^* \times E_2^*))$$

(see [18]). It follows that if $T_i \in \mathfrak{M}_{\max}(E_i^*) \cap \mathcal{S}_p$ (resp. $S_i \in \mathfrak{M}_{\max}(E_i^*) \cap \mathcal{S}_p$), $i = 1, 2$, then $T_1 \otimes T_2$ (resp. $S_1 \otimes S_2$) is a non-zero operator in $\mathfrak{M}_{\max}(\rho(E_1^* \times E_2^*)) \cap \mathcal{S}_p$ (resp. $\mathfrak{M}_{\min}(\rho(E_1^* \times E_2^*)) \cap \mathcal{S}_p$). \square

We finish this section with explicit descriptions of the spaces $\mathcal{S}_1(G)$ and $\mathcal{S}_2(G)$. For the next lemma, recall that, if G is compact then $\mathfrak{S}(G) \subseteq T(G)$ and hence $N(u) \in T(G)$ for every $u \in A(G)$.

Lemma 3.8. *Let G be a compact group. If $u \in A(G)$ then $T_{N(\tilde{u})} = \lambda(u)$.*

Proof. If $\xi, \eta \in L^2(G)$ then, using the unimodularity of G , we have

$$\begin{aligned} (T_{N(\tilde{u})}\xi, \eta) &= \iint_{G \times G} N(\tilde{u})(t, s) \xi(s) \overline{\eta(t)} ds dt \\ &= \iint_{G \times G} u(ts^{-1}) \xi(s) \overline{\eta(t)} ds dt \\ &= \iint_{G \times G} u(r) \xi(r^{-1}t) \overline{\eta(t)} dr dt = (\lambda(u)(\xi), \eta). \end{aligned}$$

□

We say that a function f belongs to $A(G)$ at the point $t \in G$ if there exists a neighbourhood U of t and a function $u \in A(G)$ such that $f(s) = u(s)$ for all $s \in U$. We let $A(G)^{\text{loc}}$ denote the set of all functions that belong to $A(G)$ at every point $t \in G$.

Lemma 3.9. *Let $w : G \times G \rightarrow \mathbb{C}$ be a measurable (with respect to product measure) function, such that, for every $r \in G$, $w(xr, yr) = w(x, y)$ for marginally almost all (x, y) . Then there exists a measurable function $u : G \rightarrow \mathbb{C}$ such that, up to a null set, $w = N(u)$.*

Proof. Let $x \in G$, and $v_x : G \rightarrow \mathbb{C}$ be given by $v_x(s) = w(s, xs)$, $s \in G$. For $r \in G$, the set

$$\{(y, z) \in G \times G : w(yr, zr) \neq w(y, z)\}$$

is marginally null. In particular,

$$\Lambda_x \stackrel{\text{def}}{=} \{(s, xs) \in G \times G : w(sr, xsr) \neq w(s, xs)\}$$

is marginally null. This easily implies that the set $\{s \in G : (s, xs) \in \Lambda_x\}$ is null and hence $v_x(sr) = v_x(s)$ for almost all s . Using arguments similar to [14, Lemma 3.2], one can prove that the function $f_x(s, r) = v_x(sr)$ is $m \times m$ -measurable. For every $r \in G$, the set $\{s \in G : v_x(sr) = v_x(s)\}$ is null. By the Fubini theorem,

$$\iint |v_x(sr) - v_x(s)| ds dr = \int \left(\int |v_x(sr) - v_x(s)| ds \right) dr = 0,$$

giving $v_x(sr) = v_x(s)$ for almost all pairs (s, r) . Thus there exists $s_0 \in G$ such that $v_x(s_0 r) = v_x(s_0)$ for almost all $r \in G$. Hence, there exists $u(x) \in \mathbb{C}$ such that $v_x(s) = u(x)$ for almost all $s \in G$.

The function $u : G \rightarrow \mathbb{C}$ is measurable as the composition of w and the measurable functions $x \rightarrow (s, xs)$. Since the functions w and $N(u)$ are equal almost everywhere on each set of the form $\{x\}^*$, applying the above arguments we have that $w = N(u)$ almost everywhere. □

Theorem 3.10. *Let G be a locally compact group. Then*

$$(3.5) \quad \mathcal{S}_1(G) = \{T \in C_r^*(G) : \text{there exists } u \in A(G)^{\text{loc}} \text{ such that}$$

$$P_K T P_K = T_{N(u)(\Delta^{-1} \otimes 1) \chi_{K \times K}} \text{ for each compact } K \subseteq G\}.$$

Moreover, if G is compact then

$$\mathcal{S}_1(G) = \{\lambda(u) : u \in A(G)\}.$$

Proof. We first show that $\mathcal{S}_1(G)$ is contained in the right hand side of (3.5). By assumption, for every compact set $K \subseteq G$ there exists a function $h_K \in T(G)$ such that $P_K T P_K = T_{h_K}$. If $L \subseteq G$ is compact then

$$\begin{aligned} T_{h_K(\chi_{K \cap L} \otimes \chi_{K \cap L})} &= P_{K \cap L} P_K T P_K P_{K \cap L} = P_{K \cap L} T P_{K \cap L} = T_{h_{K \cap L}} \\ &= P_{K \cap L} P_L T P_L P_{K \cap L} = T_{h_L(\chi_{K \cap L} \otimes \chi_{K \cap L})}. \end{aligned}$$

Thus, $h_K(\chi_{K \cap L} \otimes \chi_{K \cap L}) = h_L(\chi_{K \cap L} \otimes \chi_{K \cap L})$ almost everywhere. Since the functions $h_K(\chi_{K \cap L} \otimes \chi_{K \cap L})$ and $h_L(\chi_{K \cap L} \otimes \chi_{K \cap L})$ are ω -continuous, [17, Lemma 2.2] implies that they are equal up to a marginally null set. Let $(K_n)_{n=1}^\infty$ be an increasing sequence of compact sets such that $G = \cup_{n=1}^\infty K_n$. Setting $h(s, t) = h_{K_n}(s, t)$ if $s, t \in K_n$, we obtain a function $h : G \times G \rightarrow \mathbb{C}$, defined up to a marginally null set, which has the property that $h|_{K \times K}$ is marginally equivalent to $h_{K \times K}$ for every compact set $K \subseteq G$.

Let $L \subseteq G$ be a compact set and $\xi, \eta \in L^2(G)$ be supported on L . Fix $s \in G$ and let $M \subseteq G$ be a compact set containing both L and Ls^{-1} . Let $\rho : G \rightarrow \mathcal{B}(L^2(G))$, $s \mapsto \rho_s$, be the right regular representation given by $\rho_s \xi(x) = \sqrt{\Delta(s)} \xi(xs)$. Then

$$\begin{aligned} (T \rho_s \xi, \eta) &= (P_M T P_M \rho_s \xi, \eta) = (T_{h_M} \rho_s \xi, \eta) \\ &= \int_{G \times G} h(x, y) \xi(ys) \overline{\eta(x)} \sqrt{\Delta(s)} dx dy \\ &= \int_{G \times G} h(x, zs^{-1}) \xi(z) \overline{\eta(x)} \sqrt{\Delta(s^{-1})} dx dz. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\rho_s T \xi, \eta) &= (T \xi, \rho_{s^{-1}} \eta) = (P_M T P_M \xi, \rho_{s^{-1}} \eta) = (T_{h_M} \xi, \rho_{s^{-1}} \eta) \\ &= \int_{G \times G} h(x', z) \xi(z) \overline{\eta(x's^{-1})} \sqrt{\Delta(s^{-1})} dx' dz \\ &= \int_{G \times G} h(xs, z) \xi(z) \overline{\eta(x)} \sqrt{\Delta(s)} dx dz. \end{aligned}$$

Since $T \in C_r^*(G)$, we have $T \rho_s = \rho_s T$, and hence $\sqrt{\Delta(s^{-1})} h(x, zs^{-1}) = \sqrt{\Delta(s)} h(xs, z)$ for marginally almost all $(x, z) \in L \times L$. Since this holds for every compact set L , we have that $\sqrt{\Delta(s^{-1})} h(x, zs^{-1}) = \sqrt{\Delta(s)} h(xs, z)$ for marginally almost all $(x, z) \in G \times G$. Thus, $h(x, y) = \Delta(s) h(xs, ys)$ for marginally almost all $(x, y) \in G \times G$.

Let $\tilde{h} : G \times G \rightarrow \mathbb{C}$ be given by $\tilde{h}(x, y) = \Delta(x) h(x, y)$. If $s \in G$ then

$$\tilde{h}(xs, ys) = \Delta(xs) h(xs, ys) = \Delta(x) \Delta(s) h(xs, ys) = \Delta(x) h(x, y) = \tilde{h}(x, y),$$

for marginally almost all (x, y) . By Lemma 3.9, there exists a measurable function $u : G \rightarrow \mathbb{C}$ such that, up to a null set, $\tilde{h} = N(u)$. Thus, up to a null set, $h = (\Delta^{-1} \otimes 1) N(u)$.

Note that, for every compact set $K \subseteq G$, we have $h \chi_{K \times K} \in T(G)$ and $P_K T P_K = T_{h \chi_{K \times K}}$. Thus, it remains to show that $u \in A(G)^{\text{loc}}$. We have that $\tilde{h} \chi_{K \times K} \in T(G)$ for every compact set $K \subseteq G$. Since $T(G)$ consists of

local Schur multipliers (see [17]), the σ -compactness of G implies that \tilde{h} is a local Schur multiplier. By [18, Theorem 8.2], $u \in A(G)^{\text{loc}}$.

To see that the right hand side of (3.5) is contained in $\mathcal{S}_1(G)$, note first that for any compact set L and $u \in A(G)^{\text{loc}}$ there exists $v \in A(G)$ such that $u(t) = v(t)$ for any $t \in L$ (see the discussion before [18, Lemma 6.1]). Let K be a compact set and let $v \in A(G)$ such that $u = v$ on $L = KK^{-1}$. Then $N(u)\chi_{K \times K} = N(v)\chi_{K \times K}$. Since $N(v)$ is a Schur multiplier and $(\Delta^{-1} \otimes 1)\chi_{K \times K} \in T(G)$, we have that $h(x, y) := N(u)(x, y)\Delta^{-1}(x)\chi_{K \times K}(x, y) \in T(G)$.

If G is compact then G is unimodular and $A(G) = A(G)^{\text{loc}}$. By Lemma 3.8, $T_{N(u)} = \lambda(u)$, and the proof is complete. \square

The proof of the next proposition is similar to that of Theorem 3.10 and is omitted.

Proposition 3.11. *Let G be a locally compact group. The following are equivalent, for an operator $T \in C_r^*(G)$:*

- (i) $T \in \mathcal{S}_2(G)$;
- (ii) *there exists a measurable function $u : G \rightarrow \mathbb{C}$ such that, for every compact set $K \subseteq G$, we have $N(u)\chi_{K \times K} \in L^2(G \times G)$ and $P_K T P_K = T_{N(u)(\Delta^{-1} \otimes 1)\chi_{K \times K}}$.*

4. A p -VERSION OF THE STONE-VON NEUMANN THEOREM

The aim of this section is to establish the following p -version of the Stone-von Neumann Theorem ([22, Theorem 4.23]). We let $\mathcal{D} = \{M_a : a \in C_0(G)\}$.

Theorem 4.1. *Let G be a locally compact group and $1 \leq p \leq \infty$. Then the $\|\cdot\|_p$ -closed \mathcal{D} -bimodule generated by $\mathcal{S}_p(G)$ coincides with \mathcal{S}_p .*

Proof. For $p = \infty$, the statement reduces to the Stone-von Neumann Theorem. Fix p with $1 \leq p < \infty$. Let

$$\mathcal{S}'_p(G) = \{T \in C_r^*(G) : P_K T, T P_K \in \mathcal{S}_p, \text{ for every compact set } K \subseteq G\}$$

and

$$\mathcal{U}_p = \overline{\text{span}\{M_a T M_b : a, b \in C_c(G), T \in \mathcal{S}'_p(G)\}}^{\|\cdot\|_p}.$$

We note first that $\mathcal{S}'_p(G) \subseteq \mathcal{S}_p(G)$ and, by the definition of $\mathcal{S}_p(G)$, we have

$$\mathcal{U}_p \subseteq \overline{\text{span}\{M_a T M_b : a, b \in C_c(G), T \in \mathcal{S}_p(G)\}}^{\|\cdot\|_p} \subseteq \mathcal{S}_p.$$

It follows from the proof of Theorem 3.10, that for $u \in A(G) \cap C_c(G)$, $P_K \lambda(u)$, $\lambda(u) P_K \in \mathcal{S}_1 \subseteq \mathcal{S}_p$, $p \geq 1$ and giving that both $\mathcal{S}'_p(G)$ and \mathcal{U}_p are non-zero.

We claim that \mathcal{U}_p is an ideal of \mathcal{S}_∞ . First note that if $b \in C_c(G)$ and $f \in L^1(G)$ then by the Stone-von Neumann theorem $M_b \lambda(f)$ can be approximated in the operator norm by linear combinations $\sum_i \lambda(g_i^n) M_{c_i^n}$. Suppose that $T \in \mathcal{S}'_p(G)$, $S \in \mathcal{S}_\infty$ and $a, b, c \in C_c(G)$. Since $M_a T \in \mathcal{S}_p$ we have

$$\|M_a T M_b \lambda(f) - M_a T \sum_i \lambda(g_i^n) M_{c_i^n}\|_p \leq \|M_a T\|_p \|M_b \lambda(f) - \sum_i \lambda(g_i^n) M_{c_i^n}\| \rightarrow 0$$

and therefore $M_a T M_b \lambda(f) \in \mathcal{U}_p$. Again by the Stone-von-Neumann Theorem, S can be approximated by linear combinations of operators of the form $\lambda(f) M_c$, where $f \in L^1(G)$ and $c \in C_c(G)$. Hence the corresponding linear combination of $M_a T M_b \lambda(f) M_c$ converges in $\|\cdot\|_p$ -norm to $M_a T M_b S$. Thus, $M_a T M_b S \in \mathcal{U}_p$ and hence \mathcal{U}_p is a right ideal of \mathcal{S}_∞ ; one similarly shows that \mathcal{U}_p is a left ideal in \mathcal{S}_∞ . Thus, \mathcal{U}_p is a non-zero ideal of \mathcal{S}_∞ , closed in the Schatten p -norm. This easily implies that

$$\mathcal{U}_p = \overline{\text{span}\{M_a T M_b : a, b \in C_c(G), T \in \mathcal{S}_p(G)\}}^{\|\cdot\|_p} = \mathcal{S}_p.$$

□

5. THE CASE OF COMPACT GROUPS

In this section, we include a direct proof of Theorem 3.3, that is, a proof that does not use interpolation, in the case G is a compact group. We first recall some notions from the Fourier theory for compact groups (see, *e.g.*, [12]).

Let G be a compact group with dual \widehat{G} ; thus, \widehat{G} is a complete family of pairwise inequivalent continuous unitary representations $\pi : G \rightarrow \mathcal{B}(H_\pi)$ of G . We let $d_\pi = \dim(H_\pi)$. For $u \in A(G)$ we set

$$\hat{u}(\pi) = \int_G u(s) \pi(s^{-1}) ds,$$

understood as a linear operator on the finite-dimensional space H_π . Then

$$\sum_{\pi \in \widehat{G}} d_\pi \|\hat{u}(\pi)\|_1 < \infty,$$

where $\|\cdot\|_1$ is the trace norm. Moreover, the Fourier algebra $A(G)$ can be identified with the space of operator fields, indexed over \widehat{G} ,

$$\{(f(\pi))_{\pi \in \widehat{G}} : f(\pi) \in \mathcal{B}(H_\pi), \sum_{\pi \in \widehat{G}} d_\pi \|f(\pi)\|_1 < \infty\},$$

the identification being given by the map (which we call the Fourier transform) sending an element $u \in A(G)$ to $(\hat{u}(\pi))_{\pi \in \widehat{G}}$. Its inverse sends $(f(\pi))_{\pi \in \widehat{G}}$ to the function f given by $f(s) = \sum_{\pi \in \widehat{G}} d_\pi \text{Tr}(f(\pi) \pi(s))$ (where Tr denotes the trace). We can therefore identify the dual space of $A(G)$ with $\prod_{\pi \in \widehat{G}} \mathcal{B}(H_\pi)$ through the duality

$$(5.1) \quad \langle (T_\pi)_{\pi \in \widehat{G}}, u \rangle = \sum_{\pi \in \widehat{G}} d_\pi \text{Tr}(T_\pi \hat{u}(\pi)).$$

In particular, the evaluation functional at s corresponds to $(\pi(s))_{\pi \in \widehat{G}}$, $s \in G$. The dual space of $A(G)$ is isomorphic to $\text{VN}(G)$, and the identification of $\text{VN}(G)$ with $\prod_{\pi \in \widehat{G}} \mathcal{B}(H_\pi)$ is given by $(T_\pi)_{\pi \in \widehat{G}} \mapsto \bigoplus_{\pi \in \widehat{G}} T_\pi^{(d_\pi)} \in \text{VN}(G)$, where $T^{(k)} = \underbrace{T \oplus \cdots \oplus T}_k$. It follows from Remark (iv) at the start of Section 3

that

$$(5.2) \quad \mathcal{S}_p(G) = \left\{ T \in C_r^*(G) : \sum_{\pi \in \hat{G}} d_\pi \|T_\pi\|_p^p < \infty \right\}.$$

Thus, $\mathcal{S}_p(G)$ is an ideal not only in $C_r^*(G)$ but also in $\text{VN}(G)$.

Theorem 5.1. *Let G be a compact group, $p \geq 1$ and $E \subseteq G$ be a closed subset. The following are equivalent:*

- (i) E is an M^p -set (resp. an M_1^p -set);
- (ii) E^* is an operator M^p -set (resp. an operator M_1^p -set).

Proof. (i) \Rightarrow (ii) follows as in Theorem 3.3.

(ii) \Rightarrow (i) Suppose that E^* is an operator M^p -set. If $T \in \mathfrak{M}_{\max}(E^*) \cap \mathcal{S}_p$ is non-zero, by [18, Lemma 3.10], there exist $a, b \in L^\infty(G)$ such that $E_{a \otimes b}(T) \in C_r^*(G)$ is non-zero.

By the definition of the map $E_{a \otimes b}$ and Lemma 3.8, for $u \in A(G)$ we have

$$(5.3) \quad \begin{aligned} \langle E_{a \otimes b}(T), u \rangle &= \langle T, N(u)(a \otimes b) \rangle = \langle M_b T M_a, N(u) \rangle \\ &= \text{Tr}((M_b T M_a) T_{N(u)}) = \text{Tr}((M_b T M_a) \lambda(\check{u})). \end{aligned}$$

Let $u \in A(G)$ be such that $\sum_{\pi \in \hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q < \infty$, where q is conjugate to p . The space of all such elements is dense in $A(G)$ (indeed, it contains all elements of $A(G)$ whose Fourier transform is finitely supported). By the Peter-Weyl Theorem, $\lambda = \bigoplus_{\pi \in \hat{G}} \pi^{(d_\pi)}$, where $\pi^{(d_\pi)} = \underbrace{\pi \oplus \cdots \oplus \pi}_{d_\pi}$, and

$\lambda(\check{u})|_{H_\pi} = \hat{u}(\pi)$. Thus,

$$(5.4) \quad \|\lambda(\check{u})\|_q^q = \sum_{\pi \in \hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q < \infty.$$

By (5.3) and (5.4),

$$(5.5) \quad |\langle E_{a \otimes b}(T), u \rangle| \leq \|a\|_\infty \|b\|_\infty \|T\|_p \left(\sum_{\pi \in \hat{G}} d_\pi \|\hat{u}(\pi)\|_q^q \right)^{1/q}.$$

Let $S = E_{a \otimes b}(T)$. By (5.1), $\langle S, u \rangle = \sum_{\pi \in \hat{G}} d_\pi \text{Tr}(S_\pi \hat{u}(\pi))$. We claim that

$$(5.6) \quad \sum_{\pi \in \hat{G}} d_\pi \|S_\pi\|_p^p < \infty.$$

In fact, let $S_\pi = V_\pi |S_\pi|$ be the polar decomposition of S_π . For any finite family $\mathcal{F} \subseteq \hat{G}$, let $u \in A(G)$ be such that,

$$\hat{u}(\pi) = (|S_\pi|)^{p-1} V_\pi^* / \left(\sum_{\pi \in \mathcal{F}} d_\pi \|S_\pi\|_p^p \right)^{(p-1)/p}, \text{ if } \pi \in \mathcal{F}$$

and

$$\hat{u}(\pi) = 0, \text{ if } \pi \notin \mathcal{F}.$$

We have

$$\sum_{\pi \in \hat{G}} d_{\pi} \|\hat{u}(\pi)\|_q^q = \sum_{\pi \in \mathcal{F}} d_{\pi} \operatorname{Tr}(|S_{\pi}|^p) / \sum_{\pi \in \mathcal{F}} d_{\pi} \|S_{\pi}\|_p^p = 1$$

and

$$\begin{aligned} \sum_{\pi \in \hat{G}} d_{\pi} \operatorname{Tr}(S_{\pi} \hat{u}(\pi)) &= \sum_{\pi \in \mathcal{F}} d_{\pi} \operatorname{Tr}(|S_{\pi}|^p) / \left(\sum_{\pi \in \mathcal{F}} d_{\pi} \|S_{\pi}\|_p^p \right)^{(p-1)/p} \\ &= \left(\sum_{\pi \in \mathcal{F}} d_{\pi} \|S_{\pi}\|_p^p \right)^{1/p}. \end{aligned}$$

Inequality (5.5) now implies $(\sum_{\pi \in \mathcal{F}} d_{\pi} \|S_{\pi}\|_p^p)^{1/p} \leq \|a\|_{\infty} \|b\|_{\infty} \|T\|_p$, for any finite \mathcal{F} . Inequality (5.6) follows; by (5.2), $E_{a \otimes b}(T) \in \mathcal{S}_p(G)$. By the proof of [18, Theorem 3.11], $E_{a \otimes b}(T) \in J(E)^{\perp}$, and the proof is complete for M^p -sets. The proof of the statement for M_1^p -sets is similar and uses the fact that $E_{a \otimes b}(T) \in I(E)^{\perp}$ if $T \in \mathfrak{M}_{\min}(E^*)$. \square

Remark 5.2. If G is a compact infinite group, then $\mathcal{S}_p(G)$ is a proper ideal of $\mathcal{S}_q(G)$ if $p < q$. In fact, one can easily find $\{\alpha_{\pi}\}_{\pi \in \hat{G}} \subseteq \mathbb{C}$ such that $\sum_{\pi \in \hat{G}} d_{\pi} |\alpha_{\pi}|^p < \infty$ while $\sum_{\pi \in \hat{G}} d_{\pi} |\alpha_{\pi}|^q = \infty$. Letting now $T_{\pi} = \alpha_{\pi} P_{\pi}$, where P_{π} is a projection on a one-dimensional subspace of H_{π} , and $T = \oplus_{\pi \in \hat{G}} T_{\pi}^{(d_{\pi})}$, we have $T \in \mathcal{S}_p(G)$ but $T \notin \mathcal{S}_q(G)$. For general locally compact groups the classes $\mathcal{S}_p(G)$ may coincide, *e.g.* in the case of discrete groups, where $\mathcal{S}_p(G)$, $p \geq 1$, are all equal to $C_r^*(G)$.

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